TECHNICAL NOTES 2

PARTICLE POPULATIONS AND DISTRIBUTION FUNCTIONS

2.1   Introduction

The behavior of ore dressing equipment depends on the nature of the individual particles that are processed. The number of particles involved is very large indeed and it would be quite impossible to base computational procedures on any method that required a detailed description of the behavior of each particle. The complexity of such procedures would mean that any useful or meaningful models would be entirely out of the question. But the characteristics of individual particles do have to be taken into account and useful models cannot be developed if these are to be based simply on average properties of all the particles in the population.

Individual particles differ from each other in many respects: the differences that are of interest in ore dressing operations are those physical properties that influence the behavior of a particle when subject to treatment in any ore dressing equipment. The two most important fundamental properties are the size of the particle and its mineralogical composition. Other properties such as shape, specific gravity, fracture energy, surface area, surface energy and so on are also important and, in many ore dressing operations, can be of overriding significance. The operation of comminution and classification are primarily dependent on the size of the particles treated but the composition, specific gravity, brittleness and other properties can also influence the behavior of the particles to a greater or lesser extent during treatment. Gravity concentration operations exploit primarily the differences in specific gravity between particles and thus different mineral species can be separated from each other.

The various physical properties are not necessarily independent of each other. For example the specific gravity of a single particle is uniquely fixed once the mineralogical composition is specified. Likewise the surface energy of a particle will be specified by the mineral components that are exposed on the surface of the particle.

Some definite scheme for the description of the properties of the particles in the particle population is required that will allow sufficient detail to permit the models to be sufficiently sensitive to individual particle properties but at the same time sufficiently comprehensive to allow the economy of not having to define the properties of each individual particle. Such a scheme is provided by a description using distribution functions.

2.2   Distribution functions

The distribution function for a particular property defines quantitatively how the values of that property are distributed among the particles in the entire population. Perhaps the best known and most widely used distribution function is the particle size distribution function $P(d_p)$ defined by $P(d_p)$ = mass fraction of particles in the population with size less than or equal to d_p. The symbol d_p is used throughout these notes to represent the size of a particle.

The function $P(d_p)$ has several important general properties:
- a) $P(0) = 0$
- b) $P(\infty) = 1$
- c) $P(d_p)$ increases monotonically from 0 to 1 as d_p increases from 0 to ∞ .

Properties a) and b) are obvious because no particle in the population can have a size less than or equal to 0 and all the particles have a size less than infinity. Property c) reflects the fact that the fraction of the population having size less than or equal to d_{p1} must contain at least all those particles of size d_{p2} or smaller, if $d_{p2} \le d_{p1}$.

Of course the concept of particle size is ambiguous. Particles that are of interest in mineral processing do not have regular definable shapes such as spheres and cubes. The size of a spherical particle can be unambiguously defined as the diameter. Likewise the size of a cube can be defined unambiguously as the length of a side but another dimension could be equally well used such as the longest diagonal. Particle size clearly does not have a unique meaning even for particles with regular shapes. In mineral processing technology an indirect measure of size is used. The size of a particle is defined as the smallest hole opening in a square-mesh screen through which the particle will fall. Sometimes it is necessary to work with particles that are too small to measure size conveniently by means of screening. Then other appropriate indirect measures are used such as the terminal falling velocity in a fluid of specified viscosity and density.

In practical applications it is convenient and often essential to make use of a discrete partioning of the length scale so that the particle population is divided conceptually into groups each identified by the smallest and largest size in the group.

The value of P can be measured experimentally at a number of fixed sizes that correspond to the mesh sizes of the set of sieves that are available in the laboratory. This data is usually presented in tabular form showing mesh size against fraction smaller than that mesh. Graphical presentations are useful and are often preferred because it is generally easier to assess and compare particle size distributions when the entire distribution function is immediately visible. A variety of different graphical coordinate systems have become popular with a view to making the distribution function plot as or close to a straight line. The particle size axis is usually plotted on a logarithmic coordinate scale. The ordinate scale is calibrated according according to the distribution function $P(d_p)$ is close to a prespecified distribution. Specially ruled graph papers are available for this purpose and these can be easily drawn by computer.

The mesh sizes in the standard sieve series vary in geometric progression because experience has shown that such a classification will leave approximately equal amounts of solids on each of the test sieves in a screen analysis. Thus each mesh size is a constant factor larger than the previous one. The constant factor is usually $\sqrt[4]{2}$ or $\sqrt{2}$. The mesh sizes in such a series will plot as equidistant points on a logarithmic scale.

Although the distribution function $P(d_p)$ is perfectly well defined and is amenable to direct measurement in the laboratory, it is not directly useful for modeling ore dressing unit operations. For this purpose the derived density function is used. The discrete particle size density function $p_i(d_p)$ is defined as follows:

$$p_i(d_p) = \int_{D_i}^{D_{i-1}} dP(d_p) = P(D_{i-1}) - P(D_i) = \Delta P_i$$

$$= \text{mass fraction of the particle population}$$
$$\text{that has size between } D_{i-1} \text{ and } D_i$$

(2.1)

$p_i(d_p)$ is called the fractional discrete density function and the argument d_p is often dropped if there is no risk of confusion.

$\Delta d_p = D_{i-1} - D_i$ is the so-called size class width and is usually not constant but varies from size to size. The finite width of the size class defined by Δd_p is very important in the development of the modeling techniques that are used. The idea of a particular size class is central to the development of our modeling procedure. The size class is considered

conceptually to include all particles in the entire population that have size falling within the class boundaries d_p and $d_p + \Delta d_p$. It is customary to designate the class boundary by means of a subscript and in order to distinguish the class boundaries clearly they will always be denoted by the symbol D_i which indicates the lower boundary of size class i. Thus the entire particle population is conceptually classified into classes each one of which is defined by its upper and lower boundary. It is conventional to run the number of the classes from larger sizes to smaller. Thus $D_i \geq D_{i+1}$. The top size class has only one boundary D_1 and it includes all particles which have size greater than D_1.

The concept of the particle classes effectively allows us to formulate models for extractive metallurgical systems by describing the behaviour of classes of particles rather than the behavior of individual particles. A representative size is associated with each particle size class and it is assumed that all particles in the class will behave in our processing systems as if it had a size equal to the representative size. Clearly this will be a viable working assumption only if the size class is sufficiently narrow. It is not possible to define the concept "sufficiently narrow" precisely but it is generally assumed that a $\sqrt{2}$ series for the class boundaries is the largest geometric ratio that can be safely used. The key to the success of this approach to the modelling of particulate systems is the use of narrow size intervals. This in turn implies that a large number of particle classes must be considered. From a practical point of view this increases the amount of computation that is required if realistically precise descriptive models are to be developed for particulate processes. Consequently this approach requires efficient computer code if it is to be implemented as a viable practical tool and our emphasis is on efficient computer techniques.

2.2.1   Empirical Distribution Functions

Several empirical distribution functions have been found to represent the size distribution of many particle populations quite accurately in practice and these are useful in a number of situations. The most common are:

Rosin-Rammler distribution function defined by:

$$P(D) = 1 - \exp\left[-\left(D/D_{63.2}\right)^\alpha\right] \tag{2.2}$$

$D_{63.2}$ is the size at which the distribution function has the value 0.632.

Log-normal distribution defined by

$$P(D) = G\left(\frac{\ln(D/D_{50})}{\sigma}\right) \tag{2.3}$$

where $G(x)$ is the function

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \tag{2.4}$$

which is called the Gaussian or Normal distribution function. It is tabulated in many mathematical and statistical reference books and it is easy to obtain values for this function. In this distribution D_{50} is the particle size at which $P(D_{50}) = 0.5$. It is called the median size. σ is given by

$$\sigma = \tfrac{1}{2}(\ln D_{84} - \ln D_{16}) \tag{2.5}$$

The log-normal distribution has a particularly important theoretical significance. In 1941, the famous mathematician A N Kolmogorov[1] proved that if a particle and its progeny are broken successively and if each breakage event produces a random number of fragments having random sizes, then, if there is no preferential selection of sizes for breakage, the distribution of particle sizes will tend to the log-normal distribution after many successive fracture events. Although this theoretical analysis makes assumptions that are violated in practical comminution operations, the result indicates that particle populations that occur in practice will have size distributions that are close to log-normal. This is often found to be the case.

Logistic distribution defined by

$$P(D) = \cfrac{1}{1 + \left(\cfrac{D}{D_{50}}\right)^{-\lambda}} \tag{2.6}$$

These three distributions are two-parameter functions and they can be fitted fairly closely to measured size distributions by curve fitting techniques.

These functions have interesting geometrical properties which can be conveniently exploited in practical work.

The Rosin-Rammler distribution can be transformed to

$$\ln \ln \left(\frac{1}{1-P(D)}\right) = \alpha \ln(D) - \alpha \ln(D^*) \tag{2.7}$$

which shows that a plot of the log log reciprocal of $1\text{-}P(D)$ against the log of D will produce data points that lie on a straight line whenever the data follow the Rosin-Rammler distribution. This defines the Rosin-Rammler coordinate system.

The log-normal distribution can be transformed using the inverse function $H(G)$ of the function G. This inverse function is defined in such a way that if

$$G(x) = g \tag{2.8}$$

then

$$x = H(g) \tag{2.9}$$

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[1]Kolmogorov A N *"Uber das logarithmisch normaleVerteilungsgestz der Dimensionen der Teilchen bei Zerstuckelung". Comtes Rendus (Doklady) de l'Academie des Sciences de l'URSS* Vol 31 No 2 pp99-101, 1941. Available in English translation as "The logarithmically normal law of distribution of dimensions of particles when broken into small parts". NASA Technical Translations NASA TTF 12,287

From equation (2.3)

$$H(P(D)) = \frac{\ln(D/D_{50})}{\sigma} \qquad (2.10)$$

and a plot of $H(P(D))$ against $\log D$ will be linear whenever the data follow the log-normal distribution. This is the log-normal coordinate system.

The logistic distribution can be transformed to

$$-\log(\frac{1}{P(D)} - 1) = \lambda \log D - \lambda \log D_{50} \qquad (2.11)$$

which shows that the data will plot on a straight line in the logistic coordinate system whenever the data follow the logistic distribution. Plotting the data in these coordinate systems is a convenient method to establish which distribution function most closely describes the data.

2.2.2   Truncated size distributions

Sometimes a particle population will have every particle smaller than a definite top size. Populations of this kind occur for example when a parent particle of size D' is broken. Clearly no progeny particle can have a size larger than the parent so that the size distribution of the progeny particle population is truncated at the parent size D'. Thus

$$P(D') = 1.0 \qquad (2.12)$$

The most common truncated distribution is the logarithmic distribution
Logarithmic distribution function defined by the function:

$$P(D) = \left(\frac{D}{D'} \right)^{\alpha} \qquad \text{for } D \leq D' \qquad (2.13)$$

which clearly satisfies equation (2.12).
D' is the largest particle in the population and α is a measure of the spread in particle sizes.

Other truncated distributions are the Gaudin-Meloy and Harris distributions.

Gaudin-Meloy distribution is defined by:

$$P(D) = 1 - (1 - D/D')^n \qquad \text{for } D \leq D' \qquad (2.14)$$

Harris distribution defined by:

$$P(D) = 1 - (1 - (D/D')^s)^n \qquad for\ D \leq D' \qquad (2.15)$$

Truncated versions of the Rosin-Rammler, log-normal and logistic distributions can be generated by using a transformed size scale. The size is first normalized to the truncation size

$$\xi = D/D' \tag{2.16}$$

and the transformed size is defined by

$$\eta = \frac{\xi}{1 - \xi} \tag{2.17}$$

The truncated Rosin-Rammler distribution is

$$P(D) = 1 - \exp\left(-\left(\frac{\eta}{\eta_{63.2}}\right)^{\alpha}\right) \qquad \text{for } D \leq D' \tag{2.18}$$

The truncated log-normal distribution is

$$P(D) = G\left(\frac{\ln(\eta/\eta_{50})}{\sigma}\right) \tag{2.19}$$

with

$$\sigma = \tfrac{1}{2}(\ln(\eta_{84}) - \ln(\eta_{16})) \tag{2.20}$$

The truncated logistic distribution is

$$P(D) = \frac{1}{1 + \left(\dfrac{\eta}{\eta_{50}}\right)^{-\lambda}} \tag{2.21}$$

Straight line plots can be generated for truncated data using the appropriate coordinate systems as was done in section 2.2.

The logarithmic distribution can be transformed to

$$\log[P(D)] = \alpha \log(D) - \alpha \log(D') \tag{2.22}$$

which shows that a plot of *P(D)* against *D* on log-log coordinates will produce data points that lie on a straight line whenever the data follow the logarithmic distribution.

The Gaudin-Meloy distribution can be transformed to

$$\log(1 - P(D)) = n \log(D' - D) - n \log D' \tag{2.23}$$

Data will give a linear plot in the log-log coordinate system if plotted as 1 - *P(D)* against *D′ - D*. In order to make such a plot it is necessary to know the value of *D′* and this is a disadvantage.

The truncated Rosin-Rammler, log-normal and logistic distributions can be linearized by using the appropriate coordinate systems as described in the previous section but using the variable η in place of D. In every case, these straight line plots can be constructed only after the truncation size D′ is known.

A typical set of data measured in the laboratory is shown in Table 1.

Table 1 A typical set of data that defines the particle size distribution of a population of particles

Mesh size mm	Mass % passing
6.80	99.5
4.75	97.5
3.40	93.3
2.36	86.4
1.70	76.8
1.18	65.8
0.850	55.0
0.600	45.1
0.425	36.7
0.300	29.6
0.212	23.5
0.150	18.3
0.106	13.9
0.075	10.0
0.053	7.1
0.038	5.0

The data of table 1 is plotted in five different coordinate systems in Figure 2.1. It is useful when plotting by hand to use graph paper that is already scaled for the coordinate that is chosen. Most of the popular rulings are readily available commercially. It is even more convenient to use a computer graph plotting package such as METGRAPH which offers all of the coordinate systems shown as standard options.

Linear-linear coordinate system

Linear-log coordinate system

Log-log coordinate system

Rosin-Rammler coordinate system

Log-normal coordinate system

Logistic coordinate system
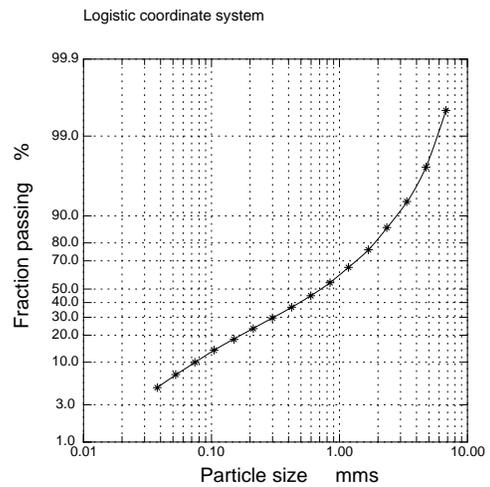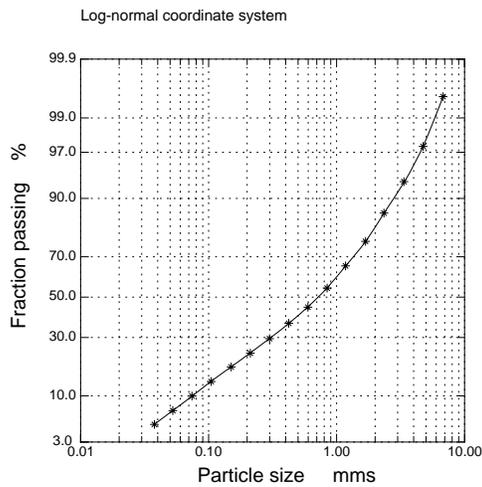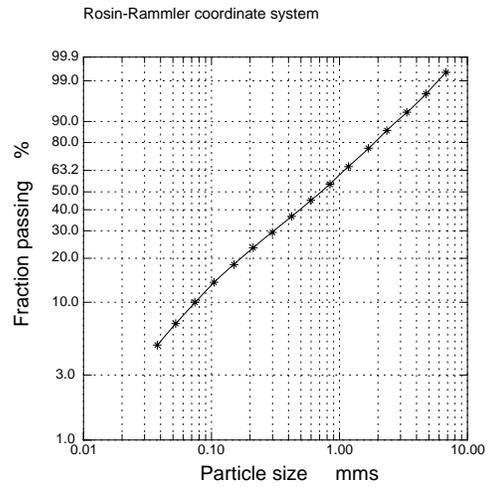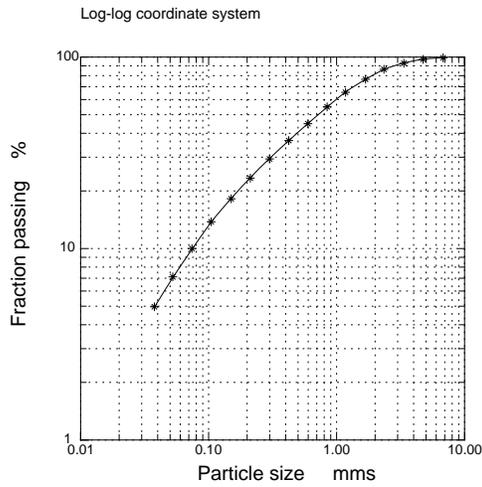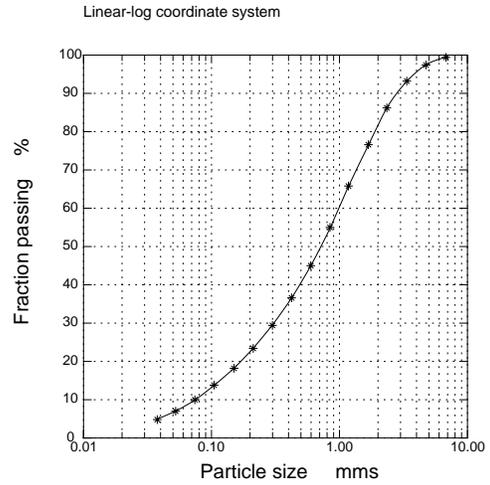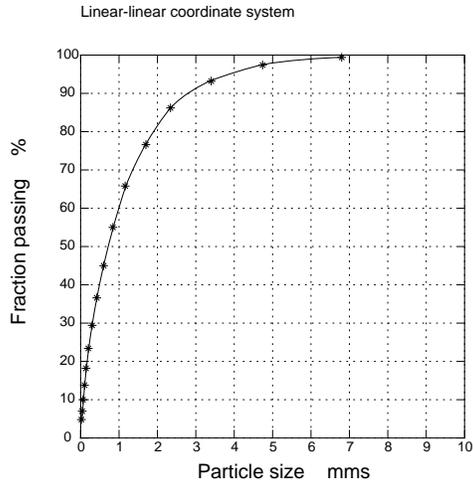
Figure 2.1 Typical particle size distribution plots showing the data of Table 1 plotted in five different coordinate systems.

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2.3   The Distribution Density Function

In much of the theoretical modeling work it will be convenient to work with a function that is derived from the distribution function by differentiation. Let x represent any particle characteristic of interest. Then $P(x)$ is the mass fraction of the particle population that consists of particles having the value of the characteristic less than or equal to x. The distribution *density* function $p(x)$ is defined by

$$p(x) = \frac{dP(x)}{dx} \tag{2.24}$$

The discrete density function defined in (2.12.1) is related to the density function by

$$p_i = \int_{D_i}^{D_{i-1}} p(x)\, dx \tag{2.25}$$
$$= P(D_{i-1}) - P(D_i)$$

A usual but imprecise interpretation of the distribution density function is that $p(x)dx$ can be regarded as the mass fraction of the particle population that consists of particles having the value of the characteristic in the narrow range $(x, x+dx)$.

An important integral relationship is

$$\int_0^\infty p(x)\, dx = P(\infty) - P(0) = 1 \tag{2.26}$$

which reflects that the sum of all fractions is unity.

2.4   The Distribution by Number, the Representative Size and Population Averages

Because all particle populations contain a finite number of particles it is also possible to describe the variation of particle characteristics through the number fraction. The number distribution function for any characteristic (having values represented by the variable x) is defined as the function $\Psi(x)$ which is the fraction by number of particles in the population having size equal to x or less. The associated number density function is defined by

$$\psi = \frac{d\Psi(x)}{dx} \tag{2.27}$$

The discrete number density is given by

$$\psi_i = \Psi\left(X_{i-1}\right) - \Psi\left(X_i\right) = \Delta\Psi_i \tag{2.28}$$

where the upper case letters represent the class boundaries.

Often it is useful to have average values for any characteristic with the average taken over all members of the population. The average value of any characteristic property is given by

$$\overline{x_N} = \frac{1}{N_T} \sum_{j=1}^{N_T} x^{(j)} \tag{2.29}$$

where $x^{(j)}$ is the value of the characteristic property for particle j and N_T is the total number of particles in the population. Equation (2.29) is unwieldy because the summation must be taken over a very large number, N_T, of particles. The number of terms in the summation is greatly reduced by collecting particles that have equal values of x into distinct groups. If the number of particles in group i is represented by $n^{(i)}$ and the value of x for particles in this group is represented by x_i, then the average value of the property x in the whole population is given by

$$\overline{x_N} = \frac{1}{N_T} \sum_{i=1}^{N} n^{(i)} x_i \tag{2.30}$$

where N represents the total number of groups that are formed. The ratio $n^{(i)}/N_T$ is the fraction by number of the particle population having size x_i. This allows an alternative and even more convenient way of evaluating the average

$$\overline{x_N} = \sum_{i=1}^{N} x_i \psi_i \tag{2.31}$$

Other averages are sometimes used. For example the average could be weighted by particle mass rather than by number

$$\overline{x} = \frac{1}{M_T} \sum_{i=1}^{N} m^{(i)} x_i \tag{2.32}$$

In equation (2.32) M_T represents the total mass of material in the population and $m^{(i)}$ the mass of particles in the group i having representative value x_i. The ratio $m^{(i)}/M_T$ is the fraction by mass of particles in the group i and this is related to the distribution function

$$\frac{m^{(i)}}{M_T} = P(x_{i+1}) - P(x_i) = \Delta P_i \tag{2.33}$$

$$\overline{x} = \sum_{i=1}^{N} x_i \Delta P_i \tag{2.34}$$

$$= \sum_{i=1}^{N} x_i p_i(x) \tag{2.35}$$

In the limit as the group widths decrease to zero this equation becomes

$$\overline{x} = \int_o^1 x \, dP(x) \tag{2.36}$$

$$= \int_o^\infty x p(x) \, dx \tag{2.37}$$

In a similar way the variance of the distribution can be obtained

$$\sigma^2 = \int_o^\infty (x-\overline{x})^2 p(x)\,dx \tag{2.38}$$

The distribution density function is useful for the evaluation of the average of any function of the particle property x.

$$\overline{f(x)} = \int_0^\infty f(x)\,p(x)\,dx \tag{2.39}$$

In the same way, the average value of the property, x, weighted by number is obtained from

$$\overline{x_N} = \int_0^\infty x\psi(x)\,dx \tag{2.40}$$

or more generally

$$\overline{f(x)_N} = \int_0^\infty f(x)\psi(x)\,dx \tag{2.41}$$

For example, if all particles in the population are spherical, the average particle volume is the average value of $\pi d_p^3/6$. Thus

$$\text{Average particle volume} = \int_0^\infty \frac{\pi d_p^3}{6}\psi(d_p)\,dd_p \tag{2.42}$$

In order that it is possible to describe the behavior of the particles adequately, the concept of a representative size for each size class is introduced. A representative size for size class i is defined through the expression

$$d_{pi}^3 = \frac{1}{\psi_i(d_p)} \int_{D_i}^{D_{i-1}} d_p^3\,\psi(d_p)\,dd_p \tag{2.43}$$

where $\psi(d_p)$ is the number distribution density function and $\psi_i(d_p)$ is the number fraction of the population in size class i. Other definitions of the representative size can be used and the precise definition will depend on the context in which the representative size will be used. It is important that the representative size be such that a single particle having the representative size would behave in a way that will adequately represent all particles in the class.

It is also possible to estimate the representative size from

$$d_{pi} = \frac{1}{p_i}\int_{D_i}^{D_{i-1}} d_p\,p(d_p)\,dd_p$$
$$= \frac{1}{p_i}\int_{D_i}^{D_{i-1}} d_p\,dP(d_p) \tag{2.44}$$

which weights the individual particles in the class by mass.

These two definitions of the representative size require the size distribution function to be known before the representative size can be established. In many circumstances this will be unsatisfactory because it would be more

convenient to have the size classes together with their representative sizes defined independently of the size distribution. A common method is to use the geometric mean of the upper and lower boundaries for the representative size.

$$d_{pi} = (D_i D_{i-1})^{1/2} \qquad (2.45)$$
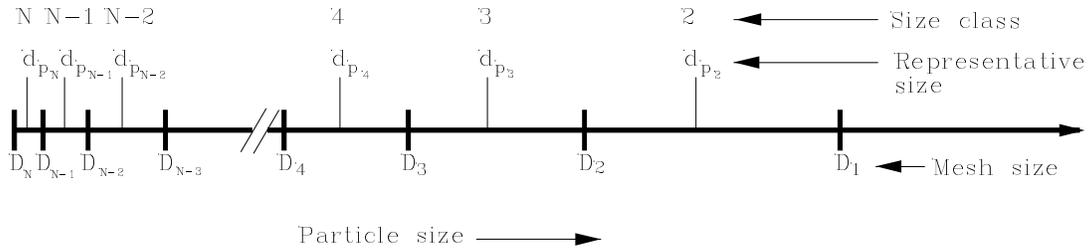
Figure 2.2 Arrangement of size classes, representative sizes and mesh sizes along the particle size axis.

Since $D_N=0$ and D_0 is undefined, equation (2.45) cannot be used to calculate the representative sizes in the two extreme size classes. These sizes are calculated using

$$d_{p1} = \frac{d_{p2}^2}{d_{p3}}$$
$$\qquad (2.46)$$
$$d_{pN} = \frac{d_{pN-1}^2}{d_{pN-2}}$$

These formulas project the sequence d_{pi} as a geometric progression into the two extreme size classes.

The arrangements of mesh and representative sizes is shown in figure 2.2

The mineralogical composition of the particles that are processed in ore dressing operations varies from particle to particle. This is of fundamental importance in any physical separation process for particulate material. The primary objective of ore-dressing processes is the separation of materials on the basis of mineralogical composition to produce concentrates having a relative abundance of the desired mineral. The objective of comminution operations is the physical separation of minerals by fragmentation. Unfortunately, except in very favorable cases, the minerals do not separate completely and many particles, no matter how finely ground, will contain a mixture of two or more mineral species. Some particles will, however, always exist that are composed of a single mineral. These are said to be perfectly liberated. The amount of mineral that is liberated is a very complex function of the crystalline structure and mineralogical texture of the ore and the interaction between these and the comminution fracture pattern.

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The mineralogical composition of a particle can be unambiguously defined by the fractional composition of the particle in terms of the individual mineral components that are of interest. Generally, more than one mineral species must be accounted for so that the mineralogical composition is described by a vector g of mineral fractions. Each element of the vector g represents the mass fraction of a corresponding mineral in the particle. The number of elements in the vector is equal to the number of minerals including gangue minerals. Thus in a particle that is made up of 25% by mass of chalcopyrite, 35% of sphalerite and 40% of gangue would be described by a mineral fraction vector g = (0,25 0,35 0,40). A number of discrete classes of mineral fractions can be defined and the range of each element fraction, i.e. the range of each component of the vector g, must be specified for each class of particles. The fractional discrete distribution function can be defined as before.

A special class exists for mineral fractions at the extreme ends of the composition range. In ore-dressing operations it is usual to work with particle populations that have some portion of the mineral completely liberated. Thus a definite non-zero fraction of the particle population can have a mineral fraction exactly equal to zero or unity and a separate class is assigned to each of these groups of particles. These classes have class widths of zero. If a single valuable mineral only is considered to be important, g is a scalar and the distribution function $P(g)$ will have the form shown in figure 2.3.
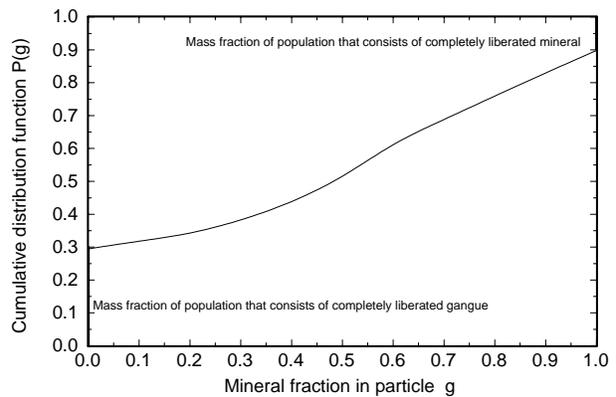
Figure 2.3 Typical particle grade distribution function for a particle population containing completely liberated and unliberated particles.

The concentration of particles in the two extreme classes representing completely liberated gangue and mineral respectively is represented by the step discontinuities in the distribution functions. When more than one mineral is significantly important, the simple graphical representation used in figure 2.3 is no longer available and a multidimensional description is required.

2.5   Joint Distribution Functions

It often happens that more than one property of the particle is significant in influencing its performance in an ore dressing operation. In that case it is essential to use a description of the particle population that takes all relevant properties into account. The appropriate description is provided by the joint distribution function. If the two properties concerned are the size and the mineralogical composition the joint distribution function would be defined as

$P(d_p,g)$ = mass fraction of the total particle population that has size $\leq d_p$ <u>and</u> mineral fraction $\leq g$.

Just as with single properties, the population can be divided into a finite number of discrete groups - each identified by a unique pair of the variables g and d_p. Thus the space of variable g and d_p can be sectioned on a rectangular grid and the discrete fractional distribution density function is defined by

$p_{ij}(d_p,g)$ = Fraction of material with size in the range (D_i,D_{i-1}), <u>and</u> composition in the range (G_{j-1},G_j)

                = mass fraction of material in area (b) in figure 2.4.

The relationship between the discrete fractional distribution density function and the two-dimensional cumulative distribution function is illustrated by reference to figure 2.4.

$P(D_{i-1}, G_j)$ = fraction of material in areas a+b+c+d

$P(D_i, G_{j-1})$ = fraction of material in area d

$P(D_i, G_j)$ = fraction of material in area a+d

$P(D_{i-1}, G_{j-1})$ = fraction of material in area c+d
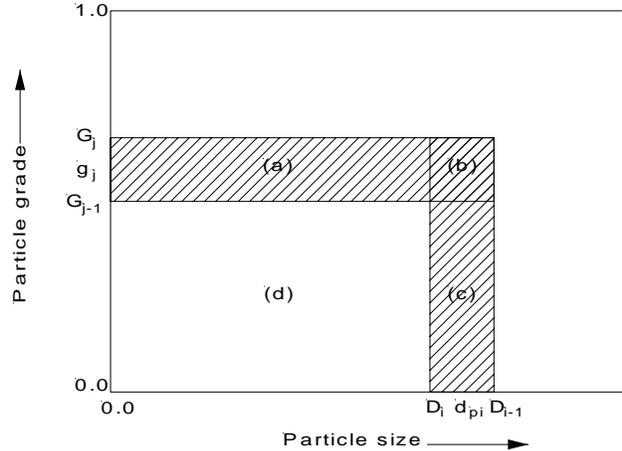
Therefore

$p_{ij}(d_p, g) = P(D_{i-1}, G_j) - P(D_i, G_j) - P(D_{i-1}, G_{j-1}) + P(D_i, G_{j-1})$

and this is the mass fraction of particle population that has

$\quad D_i \leq d_p \leq D_{i-1}$ and $G_{j-1} \leq g \leq G_j$

The ideas developed above for two-dimensional distributions are extended in the obvious way to higher dimensions. In particular, such an extension would be required for a multidimensional composition vector g if more than one mineral species is present.

It is clear from its definition that the value of $p_{ij}(d_p, g)$ does not change if the order of its arguments d_p and g, and therefore the indices i and j, is changed. Thus

$$p_{ij}(d_p, g) = p_{ji}(g, d_p) \qquad (2.47)$$

2.6   Conditional Distribution Functions

When dealing with a collection of particles it is often convenient to separate them into groups according to a single property in spite of the variability of other important properties among the particles. An obvious example of this is a classification based on size which is achieved by screening in the laboratory. The separation into size classes occurs in spite of any distribution of mineralogical composition so that particles of widely differing composition will be trapped on the same test sieve. Each batch of material on the different test sieves will have a different distribution of compositions. For example, the batch of particles in the finest size class will be relatively rich in completely liberated material. There is a unique composition distribution function for each of the size classes. The screening is called a conditioning operation and the distribution function for each size class is called a conditional fractional distribution function.

The conditional discrete density function $p_{ji}(g|d_p)$ is defined by as the mass fraction of the particles in size class i (i.e. have size between D_i and D_{i-1}), that are in composition class j. These conditional distribution functions can be related to the distribution functions that have already been defined.

2-15
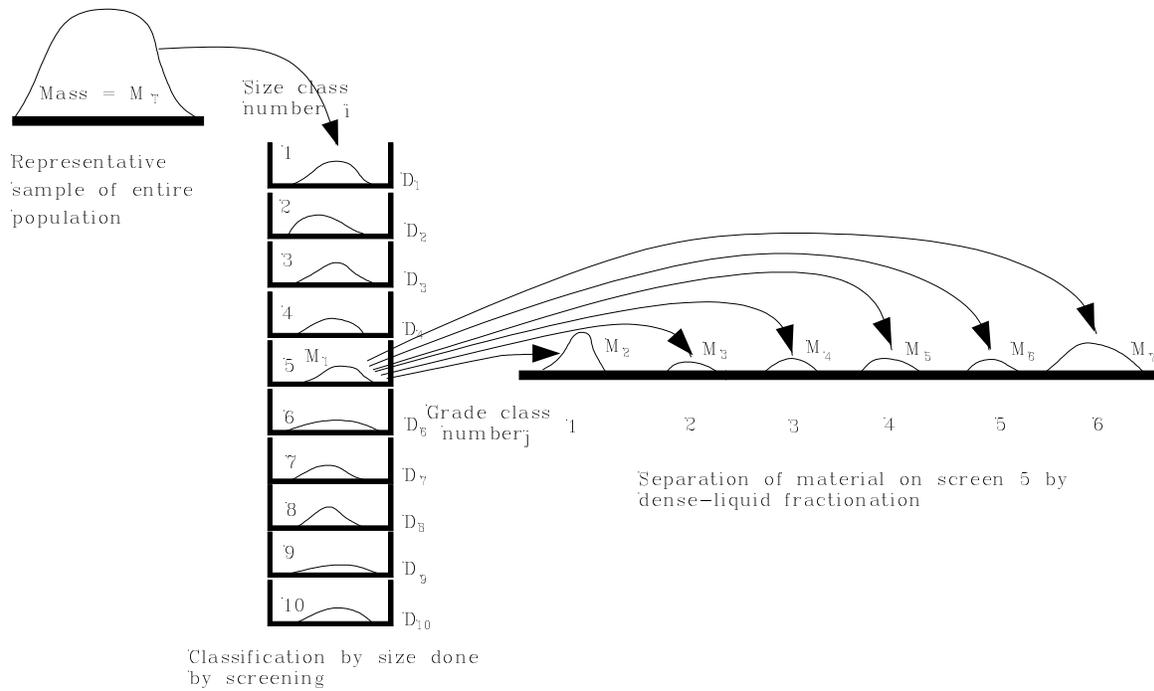
Figure 2.5 Schematic representation of the measurement of the distribution of particle grade conditioned by particle size.

The concept of the conditional distribution is illustrated schematically in Figures 2.5 and 2.6. In Figure 2.5 a representative sample of the particle population is screened and is thereby separated into the required size classes. The material on each screen is then separated on the basis of particle mineral content using, for example, dense-liquid fractionation. The two discrete distribution functions are defined in terms of the masses of material produced by these two sequential operations.

$$p_{35}(g, d_p) = \frac{M_4}{M_T} \tag{2.48}$$

$$p_{35}(g \mid d_p) = \frac{M_4}{M_1} \tag{2.49}$$

and

$$p_5 (d_p) = \frac{M_1}{M_T} \tag{2.50}$$

2-16

It is easy to see that
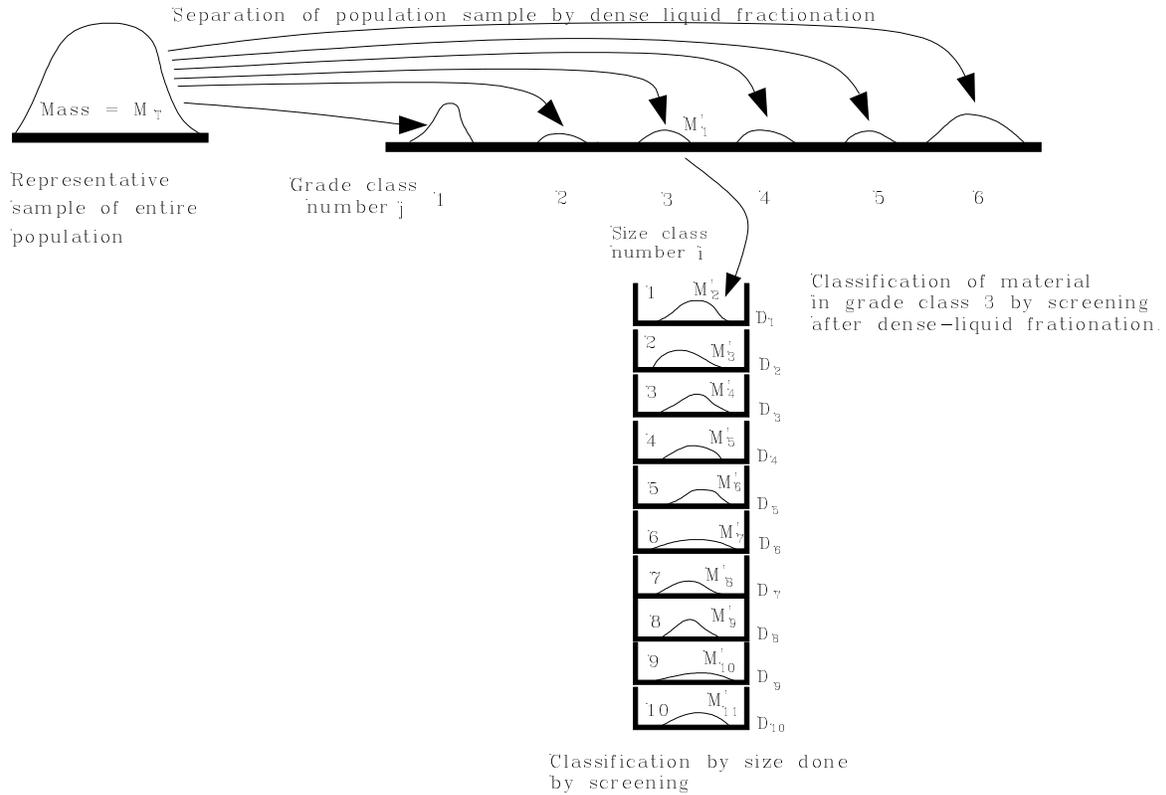
Figure 2.6 Schematic representation of the measurement of the distribution of particle size condtioned by particle grade.

$$p_{35}(g|d_p) \; = \; \frac{M_4}{M_1} \; = \; \frac{M_4/M_T}{M_1/M_T} \; = \; \frac{p_{35}(g,d_p)}{p_5(d_p)} \tag{2.51}$$

and that $p_{j5}(g|d_p)$ shows how the material on screen 5 is distributed with respect to the particle composition.

In Figure 2.6 the separation is done on the basis of mineral content first, then each particle grade class is classified by screening. This produces the size distribution conditioned by particle grade.

$$p_{53}(d_p|g) \; = \; \frac{M_6{'}}{M_1{'}} \; = \; \frac{M_6{'}/M_T}{M_1{'}/M_T} \; = \; \frac{p_{53}(d_p,g)}{p_3(g)} \tag{2.53}$$

It is clear that

$$M_2 + M_3 + M_4 + M_5 + M_6 + M_7 = M_1 \tag{2.54}$$

so that

$$\sum_{j=1}^{6} p_{ji}(g|d_p) = 1 \tag{2.55}$$

and

$$\sum_{j=1}^{6} p_{j5}(g,d_p) = p_5(d_p) \tag{2.56}$$

These ideas can be generalized to develop the following relationships. If M is the mass of the total population, the mass of particles that fall in the two classes j and i simultaneously is just $Mp_{ji}(g,d_p)$. When this is expressed as a fraction of only those particles in the d_p class, namely $Mp_i(d_p)$ the conditional distribution is generated.

Thus

$$\begin{aligned} p_{ji}(g|d_p) &= \frac{Mp_{ji}(g,d_p)}{Mp_i(d_p)} \\ &= \frac{p_{ji}(g,d_p)}{p_i(d_p)} \end{aligned} \tag{2.57}$$

Equation (2.57) is important chiefly because it provides a means for the determination of the theoretically important joint discrete distribution function $p_{ji}(g,d_p)$ from the experimentally observable conditional distribution function $p_{ji}(g|d_p)$

$$p_{ji}(g,d_p) = p_{ji}(g|d_p)p_i(d_p) \tag{2.58}$$

Since

$$p_{ji}(g,d_p) = p_{ij}(d_p,g) \tag{2.59}$$

we note that

$$p_{ji}(g,d_p) = p_{ij}(d_p|g)p_j(g) \tag{2.60}$$

and

$$p_{ji}(g|d_p)p_i(d_p) = p_{ij}(d_p|g)p_j(g) \tag{2.61}$$

Equation (2.58) corresponds to an experimental procedure in which the particle population is first separated on the basis of size by screening followed by a separation of each screened fraction into various composition groups. Equation (2.61) on the other hand corresponds to a separation on the basis of composition (by magnetic, electrostatic, or dense liquid techniques perhaps) followed by a sieve analysis on each composition class. Either way the same joint distribution function is generated but the former experimental procedure is in most instances much less convenient than the latter because of the experimental difficulties associated with separation by composition. It is usually more efficient

to combine one composition separation with many size separations (which are comparatively simple to do in the laboratory) than the other way about.

The density functions satisfy the following general relationships which may be easily verified using the same simple principles.

$$\sum_i \sum_j p_{ij}(x,y) = 1 \tag{2.62}$$

$$\sum_j p_{ij}(x,y) = \sum_j p_{ij}(x|y)p_j(y) = p_i(x) \tag{2.63}$$

$$\sum_i p_{ij}(x,y) = \sum_i p_{ji}(y|x)p_i(x) = p_j(y) \tag{2.64}$$

$$\sum_i p_{ij}(x|y) = 1 \tag{2.65}$$

The principles developed in this section can be used to define the conditional distribution functions $P(g|d_p)$ and $P(g|d_p)$ as well as the associated density functions $p(g|d_p)$. These are related by

$$p(x|y) = \frac{dP(x|y)}{dx} \tag{2.66}$$

$$p(x|y) = \frac{p(x,y)}{p(y)} \tag{2.67}$$

and satisfy the following relationships analogous to equations (2.62) to (2.65)

$$\int \int p(x,y)\,dx\,dy = 1 \tag{2.68}$$

$$\int p(x,y)\,dy = p(x) \tag{2.69}$$

$$\int p(x,y)\,dx = \int p(y|x)p(x)\,dx = p(y) \tag{2.70}$$

$$\int p(x|y)\,dx = 1 \tag{2.71}$$

2.6.1 Practical representations of conditional grade distributions - The washability curve

Conditional grade distributions have been used for many years in practical mineral processing and a number of standard representational methods have evolved. Of these, the most widely used is the washability distribution and the associated washability curve. This method was developed initially to analyze coal washing operations and it is based on the dense-liquid fractionation laboratory method. This procedure is based on the use of a sequence of organic liquids having different densities usually in the range 1200 kg/m^3 to about 3200 kg/m^3 although denser liquids can be synthesized and used. The fractionation method relies on careful separation of the floating and sinking fractions when a representative sample of the test particle population is allowed to separate in a quiescent liquid of specified density. The fraction that floats represents the fraction of the particle population that has density less than the density of the test liquid. If the composition of the particles can be related directly to the density of the particle, the measured fraction is equal to the cumulative distribution $P(g)$. It is common practice to undertake dense-liquid fraction testing on specific size fractions in which case the conditional cumulative grade distribution $P(g|d_{pi})$ is generated in the experiment. The value of the
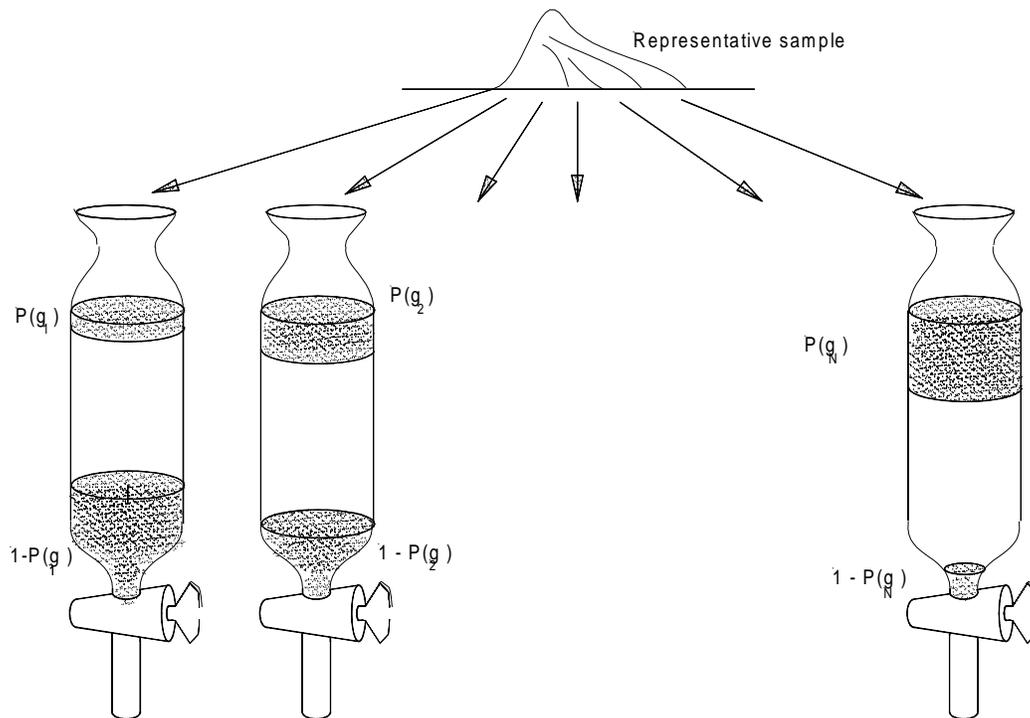
Figure 2.7 *Representation of the dense-liquid fractionation experiments. The parallel method generates the cumulative distribution P(g). If a particular size fraction is used, this test generates the conditional distribution $P(g|d_p)$*

cumulative distribution function can be measured at different values of g by using a series of liquids whose densities are adjusted to correspond to specific particle grades although it is more common to set up the liquid densities on a regular pattern to suit the material under test. The dense-liquid fractionation test is illustrated in Figure 2.7. Because identical representative samples of the particle population are analyzed in parallel, this method of analysis is called the parallel method.
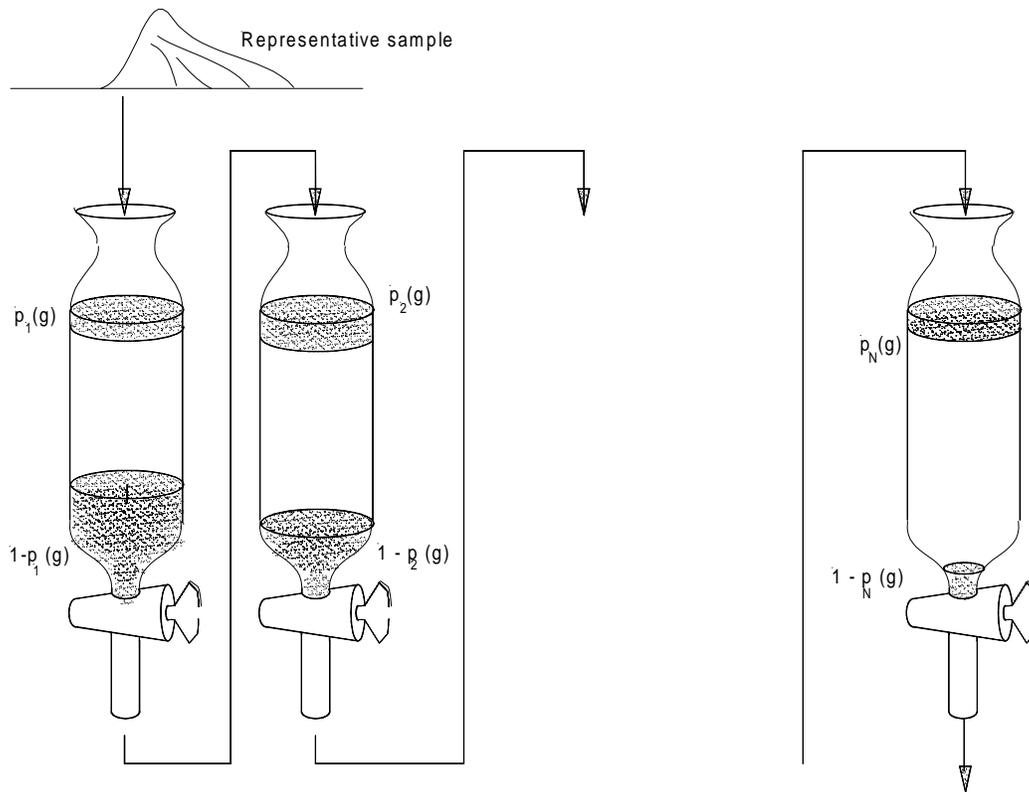
Figure 2.8 *Representation of the dense-liquid fractionation experiment. The sequential method generates the discrete distribution $p_j(g)$. If a particular size fraction is used, this test generates the conditional discrete distribution $p_j(g_j|d_p)$*

An alternative method for this analysis is often used because it uses smaller quantities of sample and because it yields additional information that is particularly useful and valuable in the analysis and simulation of mineral processing operations. Instead analyzing *N* identical samples at *N* different densities, a single sample is separated sequentially at N different densities. The sink fraction from the first test is tested at the next higher density after which the sink from the second test passes to the third liquid and so on until separations at all *N* densities have been completed. This method is illustrated in Figure 2.8

Each float fraction is collected and, after removal of any adhering liquid, is available for further analysis. Additional analyses that can be done include average particle density by pycnometry, elemental analysis by X-ray fluorescence analysis or atomic adsorption spectrometry and particle size analysis if a size-composite sample is used.

A typical set of data from a dense liquid test is shown in Table 2.

Table 2.1 Typical data from a dense-liquid test.

Liquid specific gravity	Total mass yield in this fraction %	% CaO	% SiO$_2$
Float at 2.85	21.6	19.30	18.2
2.85 - 2.88	5.70	21.76	2.49
2.88 - 2.91	3.20	10.15	1.52
2.91 - 2.94	0.90	9.67	2.92
2.94 - 2.96	7.60	2.95	3.89
2.96 - 3.03	61.0	0.96	2.55
Sink at 3.03	0.00		

In order to convert the experimental data obtained in the dense-liquid test to the distribution with respect to particle composition it is necessary to relate the density of a particle to its mineralogical composition. If the material is a simple two-component mineral mixture this is

$$\frac{1}{\rho} = \frac{g}{\rho_M} + \frac{1 - g}{\rho_G} \tag{2.72}$$

where ρ = density of a particle

ρ_M = density of the mineral phase

ρ_G = density of the gungue phase

g = mass fraction of mineral in the particle

The inverse of this equation is more useful

$$g = \frac{\rho_G - \rho}{\rho_G - \rho_M} \frac{\rho_M}{\rho} \tag{2.73}$$

which shows that the mineral grade is a linear function of the reciprocal of the particle density.

When the mineralogical texture is more complex than the simple binary mixture of two minerals, additional information is required from the dense-medium test to relate the particle composition to the separating density. This requires either elemental analysis of the individual fractions that are obtained in the sequential dense-liquid test. Typical data are shown in Table 2.1 From the measured assays, the average mineralogical composition of particles in each fraction can be estimated. In this case the calcite content is estimated from the CaO assay and the magnesite content is estimated by difference assuming that only the three minerals magnesite, calcite and silica are present. The relationship between the particle density and its mineralogical composition is

$$\rho^{-1} = \sum_{m=1}^{M} g_m \rho_m^{-1} \tag{2.74}$$

If the density intervals used in the dense-liquid test are narrow it is reasonable to postulate that the average density of the particles in each density fraction is the midpoint between the end points of the intervals.

The densities calculated from the mineralogical compositions and the known densities of the minerals should correspond quite closely with the midpoint densities as shown in Table 2.2.

Table 2.2 Data derived from the measured data in table 2.1.

Liquid specific gravity	% Magnesite	% Calcite	% SiO_2	Calculated density kg/m^3
Float at 2.85	47.34	34.46	18.2	2828
2.85 - 2.88	58.65	38.86	2.49	2867
2.88 - 2.91	80.36	18.13	1.52	2935
2.91 - 2.94	79.81	17.28	2.92	2933
2.94 - 2.96	90.85	5.26	3.89	2968
2.96 - 3.03	95.74	1.71	2.55	2985
Sink at 3.03				

In the case of coal, it is usual to measure the ash content and sulfur content of the washability fractions. It is also customary to measure the energy content of the fractions as well because this plays a major role in assessing the utility of the coal for power generation. Even greater detail regarding the make up of the coal can be obtained if the full proximate analysis is determined for each washability fraction. It is also possible to distinguish organically bound sulfur from pyritic sulfur in the coal. The greater the detail of the analysis of the washability fractions the greater the detail in the products that can be calculated by modeling and simulation.

2.6.2 Measurement of grade distributions by image analysis

In recent years a more direct method of measurement has been developed namely mineral liberation measurement using automatic image analysis. This technique provides a direct measurement of the distribution of particle grades in a sample from a narrow size fraction. The technique requires the generation of microscopic images of the particles that are mounted in random orientation and sectioned. The images which can be generated by optical or scanning electron microscopy must define each mineral phase that is to be measured. A typical image of a binary mineral system is shown in Figure 2.9. The apparent grade of each particle section in the image can be readily determined when the image is stored in digital form. The apparent areal grad of a particle section is the ratio of mineral phase pixels to total pixels in the section.
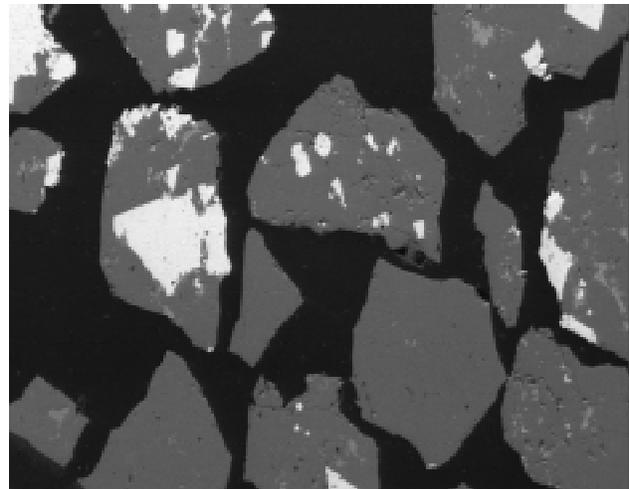
Figure 2.9 *Electron microscope image of mineral particle sections showing separate phases.*

Alternatively the apparent grade of many linear intercepts across a particle section can be measured. It is a simple matter to establish the distribution of apparent linear or areal grades from images containing a sufficiently large number of particle sections. Typical histograms of measured linear grade distributions are shown in Figure 2.10. These histograms are based on 12 grade classes which describe pure gangue in class 1, pure mineral in class 12 and 10 equally spaced grade classes in the grade fraction range from 0.0 to 1.0. Note that 11 separate size fractions were analyzed.
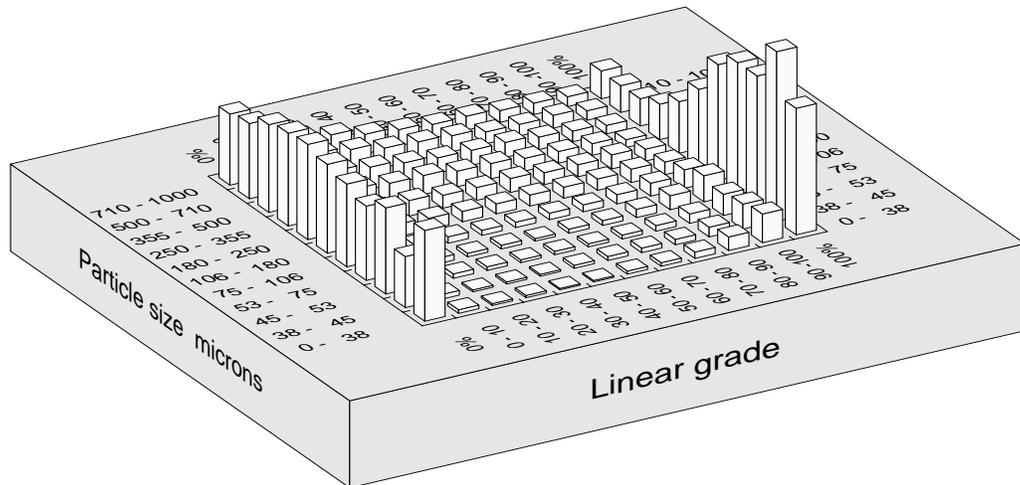
Figure 2.10 *Distribution of linear grades measured by image analysis of many images such as that shown in Figure 2.9. 11 separate size fractions were analyzed.*

The measured apparent grade distributions must be stereologically corrected to convert them to the desired grade distribution. The stereological correction is a typical inverse problem and requires the solution of the integral equation
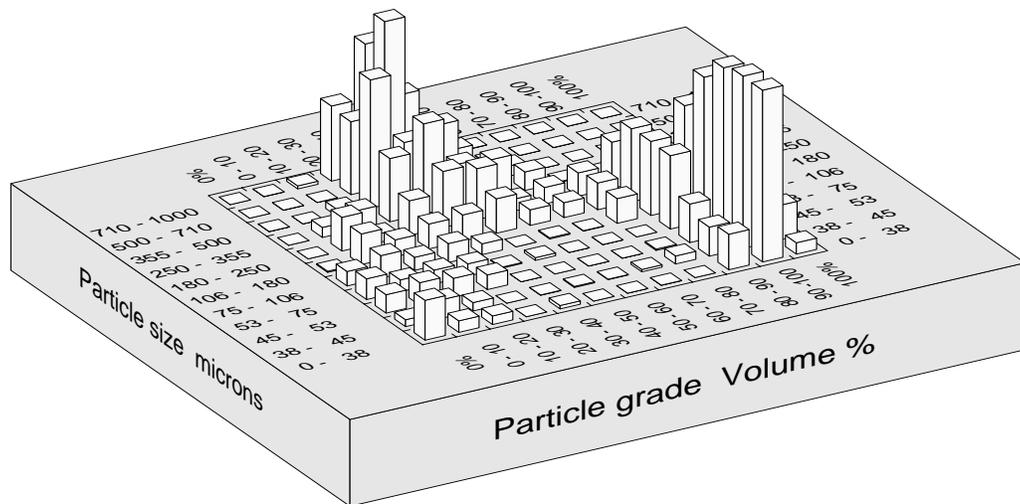
Figure 2.11 *Liberation distributions of 11 particle size fractions of a 2-component ore measured by image analysis. These histograms were obtained by stereological correction of the data shown in Figure2.10*

$$P(g_m|d_p) = \int_0^1 P(g_m|g,d_p)\, p(g|d_p)\, dg \tag{2.75}$$

where g_m represents the measured apparent grade, either linear or areal, and g represents the true grade of a particle. $P(g_m|dp_i)$ is the cumulative distribution of apparent grades that is measured in the image. Solution of the integral equation requires considerable care to ensure reliable answers but effective methods are readily available. A histogram of the true volumetric distribution of particle grades is shown in Figure 2.11 after stereological correction of the data in Figure 2.10.

2.7   Independence

It happens that sometimes two properties can be distributed independently of each other. This idea can be made precise by defining the independence of two properties, say k and d_p, if the following relationship is satisfied.

$$p_{ji}(k|d_p) = p_j(k) \tag{2.76}$$

which means that the distribution of k values (these could be flotation rate constants for example) is the same within any size class as it is across the entire population.

This leads to

$$\begin{aligned} p_{ji}(k,d_p) &= p_j(k|d_p)p_i(d_p) \\ &= p_j(k)p_i(d_p) \end{aligned} \tag{2.77}$$

which shows that the joint distribution for two properties that are independent can be generated as the product of the two separate distribution functions.

2.8   Distributions by Number

In some situations it is useful to use number fractions rather than mass fractions when dealing with particle populations. The relationship between the mass distribution functions and the equivalent number distribution function can be deduced as follows.

The number distribution function $\Psi(d_p)$ is defined to be equal to the number fraction of particles in the entire population with size $\leq d_p$. Number distribution functions and number distribution density functions can be defined in a similar way for each of the distribution types already defined for mass fractions. In particular the discrete number fractional distribution function is defined by

$$\psi_i(d_p) = \frac{\text{number of particles in size class i}}{\text{Total number of particles in the population}}$$

$$= \frac{n_i(d_p)}{N}$$

(2.78)

The number distributions can be related to the mass distributions as follows

Let $\psi(m)dm$ = number fraction of particles having mass in $(m,m+dm)$.

$\psi(m|d_p)dm$ =

number fraction of particles of size d_p that have mass in $(m,m+dm)$. That is the distribution density for particle mass conditioned by particle size.

$p_i(d_p)$ = mass fraction of particles in size class i.

m = Mass of a particle of size d_p.

$\bar{m}(d_p)$ = average mass of a particle of size d_p.

M = total mass of particles in the population.

N = total number of particles in the population.

$$M\ p(d_p) = N\int_o^\infty m\ \psi(m,d_p)\,dm$$

$$= N\int_o^\infty m\ \psi(m|d_p)\psi(d_p)\,dm$$

(2.79)

$$= N\psi(d_p)\int_o^\infty m\psi(m|d_p)\,dm$$

$$= N\psi(d_p)\bar{m}(d_p)$$

Using (2.25)

$$Mp_i(d_p) = M\int_{D_i}^{D_{i-1}} p(d_p)\,dd_p$$

(2.80)

$$Mp_i(d_p) = N\int_{D_i}^{D_{i-1}} \psi(d_p)\bar{m}(d_p)\,dd_p$$

If the shape of a particle is statistically independent of size (a good approximation for single particle populations) then

$$\beta = \frac{\bar{m}(d_p)}{d_p^3} = \frac{\bar{m}(d_{pi})}{d_{pi}^3}$$

(2.81)

is constant independent of size.

Equation (2.80) can be simplified by using equation (2.81) together with equation (2.43)

$$Mp_i(d_p) = N\frac{\overline{m}(d_{pi})}{d_{pi}^3}\int_{D_i}^{D_{i-1}}d_p^3\psi(d_p)dd_p$$

$$= N\overline{m}(d_{pi})\psi_i(d_p)$$

(2.82)

The relationship between N and M is obtained from

$$M = M\sum_i p_i(d_p) = N\sum_i \overline{m}(d_{pi})\psi_i(d_p)$$

$$\frac{M}{N} = \sum_i \overline{m}(d_{pi})\psi_i(d_p)$$

(2.83)

$$= \beta\sum_i d_{pi}^3\psi_i(d_p)$$

or

$$\frac{M}{N} = \beta\int_o^\infty d_p^3\psi(d_p)dd_p$$

(2.84)

where equation (2.81) has been used.
Substituting equation (2.84) into equation (2.82) gives an exact relationship between the discrete distribution by mass $p_i(d_p)$ and the discrete distribution by number $\psi_i(d_p)$

$$p_i(d_p) = \frac{d_{pi}^3\psi_i(d_p)}{\sum_i d_{pi}^3\psi_i(d_p)}$$

(2.85)

Similarly

$$\psi_i(d_p) = \frac{p_i(d_p)/d_{pi}^3}{\sum_i p_i(d_p)/d_{pi}^3}$$

(2.86)

2.9   Internal and External Particle Coordinate and Distribution Densities

Up to now, the particles have been classified according to two very important properties: the particle size and the mineralogical composition. These are two examples of many properties that can be used to describe the particle. They are certainly the most important descriptive properties so far as ore-dressing and hydrometallurgical processes are concerned but in order to develop effective models for the unit operations it is always necessary to ensure that the particles are described in sufficient detail for the purposes of the modelling study. It is impossible to make a complete description of any one particle - such concepts as particle shapes, surface topography, surface energy etc. cannot be

completely described in a quantitative sense using a finite number of variables. Consequently, it is always necessary to choose a finite and comparatively small number of variables that can be used to describe the particle properties in sufficient detail for the purposes of the model.

The properties that describe the nature of the individual particles are called the internal coordinates of the particle phase space. The choice of these coordinates is dictated entirely by the demands of the model. However, it is always necessary to bear in mind that one of the primary objectives of modeling the unit operations is to link these together in a plant flowsheet so that the plant can be simulated. It will often happen that the set of internal coordinates that are required to model one of the unit operations might not be appropriate for the other models. In general the problem is overcome by specifying a set of internal coordinates that will include all internal coordinate required for the separate models as subsets. This means that some of the internal coordinates will be redundant in some of the unit models. This does not cause any problems in general.

In some unit operations, the physical location of the particle in the unit equipment can have a very significant effect on the particle behavior and it is sometimes necessary to track the position of the particles in the equipment in order to describe the operation of the unit as a whole. This is not always necessary but in formulating the general model structure it is convenient to include them.

The distribution functions that were described in the previous section can be related to corresponding distribution density functions and it is convenient to formulate a generalized model in terms of the joint distribution density function for all the particle coordinates external and internal.

The generalized particulate distribution density function $\psi(x)$ is an ordinary function of the vector argument x which comprises all of the external and internal coordinates that are relevant to the problem on hand. All possible vectors x make up the particle phase space. $\psi(x)$ is defined as the number fraction density i.e. the number fraction of the entire particle population that occupies unit volume of the particle phase space. The finite particle distribution functions defined previously can be constructed from the distribution density as follows.

In the case where the phase space consists only of the particle size (d_p) and the particle grade (g)

$$\Psi_{ij}(d_p, g_i) = \int_{D_i}^{D_{i-1}} \int_{G_i}^{G_{i+1}} \psi(d_p, g) \, dd_p \, dg \qquad (2.87)$$

The mathematical description of ψ is sometimes difficult. For example it may not be possible to obtain ψ in terms of any known transcendental functions and ψ may have to include Dirac delta functions. Particular examples are the two extreme ends of the liberation spectrum. Thus we tend to avoid ψ for practical modeling work and use it only formally to develop some necessary model structures.

2.10   Particle Properties Derived from Internal Coordinates

The internal coordinates of a particle should be sufficient to describe all significant characteristics of the particle. In many situations it is necessary to obtain some derived quantity. Some examples are: the metal assay from the mineralogical composition, the magnetic susceptibility of the particle from the magnetic susceptibilities of the individual

minerals, and the calorific value, volatile content and ash content of a coal particle from the mineral matter content plus the maceral content.

The density of the particle is a particularly important derived property and we note that this is related to mineralogical composition by

$$\frac{1}{\rho} = \sum_{m=1}^{M} \frac{g_m}{\rho_m} \tag{2.88}$$

where

ρ_m = density of mineral phase.

ρ = density of particle.

g_m = mass fraction of mineral phase m in the particle (particle grade).

2.11   The Population Balance Modeling Method

The characterization of a particle population through the distribution of particles in the external and internal variable spaces provides a very powerful formalized modeling procedure. The essential idea is that as the particles move through the processing environment their external and internal coordinates change - the external coordinates because they move physically and the internal coordinates because the particles are altered by the processing.

The collection of external and internal coordinates attached to a particle defines the coordinate location of the particle in the phase space. The external coordinates describe its actual physical location while the internal coordinates describe the characteristics of the particle itself. The internal coordinates must be sufficiently numerous to describe the particle in all the detail necessary for its complete characterization in terms of the processes that must be described by the model.

Some examples of internal coordinates are
- Particles size; this is perhaps the most important of all
- mineralogical composition
- chemical composition
- particle shape
- surface specific energy

Some of these internal coordinates may themselves be multidimensional. For example the mineralogical composition of the solid material may include more than two minerals in which case the mineralogical composition vector must include one coordinate for each distinct mineral. Sometimes the multidimensionality of the internal coordinates must be infinite. This occurs, for example, during leaching processes when the concentration of the species that is being leached varies continuously within the particle and the chemical composition must be known at every point within the particle. In general the concentration profile of the leached component will be low near the surface of the particle and higher further in where lixiviant has not attained high concentrations because of diffusional resistance within the particle. The concentration profile is a functional of the entire processing history of the particle. In practice it is not convenient to use an infinite-dimensional internal coordinate and various finite-dimensions approximations must be made. A very important instance when a one-dimensional internal characterisation is completely adequate applies to

the technically important topochemical processes where the radial position of the topochemical interface inside the particle often carries complete information regarding the concentration profile and its effect on the rate of the chemical process. There is a large class of particulate rate processes that can be modeled topochemically and for which the instantaneous rate of the process over the particle as a whole can be written as a unique function of the position of the topochemical interface. This makes topochemical models very powerful models indeed for the descriptions of particulate rate processes. Furthermore topochemical models permit the calculation of the rate at which the topochemical boundary moves and this rate is very important in the formulation of population balance models for the particle population as a whole.

2.12 The Fundamental Population Balance Equation

Some definitions

$\psi(x)dx$ is the number fraction of particles per unit volume of phase space. i.e. the number fraction of the particle population that occupies a small volume around the phase coordinate x.

x is the coordinate point in phase space $x \varepsilon R^N$

u is a vector of "velocities" at which particles change their phase coordinates. For example this could be the rate of change of particle size by attrition when particle size is an internal coordinate, the rate of movement of the topochemical interface when the position of the interface is the internal coordinate or the rate of change of particle composition when the particle undergoes chemical change. The corresponding elements for the external coordinate are the physical velocities in three-dimensional space.

$R(x)$ is the rate at which particles at coordinate position x are destroyed. This rate is specified as mass per unit volume of phase space per unit time.

W_{in} is the mass rate at which solid material is fed to the system.

W_{out} is the mass rate at which solid material is withdrawn from the system.
    The material in the feed and product is distributed in the phase space according to ψ_{in} and ψ_{out} respectively.

$\bar{m}(x)$ is the average mass of a particle at point x in the phase space.

N is the total number of particles in the system.

A is the rate of addition of material in the feed stream.

Q is the rate of removal material through product removal streams.

$B(x;x')$ is a distribution function that describes the way material can move suddenly over discrete distances in phase space following physical destruction or breakage. $B(x;x')$ is the mass fraction of solid material that has a phase coordinate less then x after destruction of a particle at phase coordinate x'. The statement that the phase coordinate of a particle is less than x must be interpreted to mean that every phase coordinate of the particle is less than the corresponding element of x. $b(x;x')$ is the corresponding density function.

$a(x;x')$ is the distribution density for particles produced by attrition and other surface wear processes at x'.

In general $a(x;x')$ and $b(x;x')$ are very different from each other and in practical applications each needs to be determined separately.

Consider a region R_c of the particle phase space and account for the accumulation of particulate mass in R_c by all processes that change the coordinates of particles. The region R_c is completely enclosed by a surface S_c.

These processes are
1. Movement across the boundaries of R_c (convective motion in phase space).
2. Arrivals by finite steps from other regions in the phase space.
3. Destruction of particles in region R_c.
4. Physical additions and withdrawals in feed and product streams.

An accounting of individual particles in the reference region R_c gives

$$\frac{\partial}{\partial t} \int_{R_c} N\psi(\boldsymbol{x})\, d\boldsymbol{x} = -\int_{S_c} N\psi(\boldsymbol{x}) \boldsymbol{u.n}\ d\sigma - D + B - Q + A \tag{2.89}$$

In this equation, $\boldsymbol{n}$ is the outward pointing normal vector to the surface S_c at point **x.**

In our mineral processing systems solid mass is conserved and this imposes some important constraints on the formulation of model equations.

The destruction processes can generally be described by a mass rate of destruction per unit volume of phase space

$$D = \int_{R_c} \frac{R(\psi(\boldsymbol{x}),\boldsymbol{x},F[\psi(\boldsymbol{x})])}{\overline{m}(\boldsymbol{x})}\, d\boldsymbol{x} \tag{2.90}$$

and is specified as number of particles broken per unit time in the control volume R_c. The notation $F_1[\psi(\boldsymbol{x})]$ indicates that the destruction function process D is a function of the entire distribution function $\psi(\boldsymbol{x})$ and not only of the value of $\psi(\boldsymbol{x})$ at **x**. This allows for such effects as the variation of breakage rates in a mill with the amount of fine and coarse particles present.

The birth processes can only result (and must necessarily do so) from the products of the destruction processes since mass is conserved.

$$B = \int_{R_c} \frac{1}{\overline{m}(\boldsymbol{x})} \int_{R'(\boldsymbol{x})} R(\psi(\boldsymbol{x}'),\boldsymbol{x}',F[\psi(\boldsymbol{x})])\, b(\boldsymbol{x};\boldsymbol{x}')\, d\boldsymbol{x}'\, d\boldsymbol{x}$$

$$- \int_{R_c} \frac{N}{\overline{m}(\boldsymbol{x})} \int_{R''(\boldsymbol{x})} \psi(\boldsymbol{x}')\, \boldsymbol{u}(\boldsymbol{x}').\nabla \overline{m}(\boldsymbol{x}')\, a(\boldsymbol{x};\boldsymbol{x}')\, d\boldsymbol{x}'\, d\boldsymbol{x} \tag{2.91}$$

Here $R'(\boldsymbol{x})$ and $R''(\boldsymbol{x})$ are the regions of space from which progeny particles can enter the small phase volume $d\boldsymbol{x}$ around the point $\boldsymbol{x}$ by breakage and attrition processes respectively. These are called the feeder regions for point $\boldsymbol{x}$.

The conservation of mass constraint distinguishes mineral processing systems from other particulate processes such as crystallization and leaching in which mass transfers between the solid phase and the liquid phase and solid phase mass is not conserved.

Mass enters and leaves the processing volume by physical addition and removal through feed and product streams. The addition rate is given by

$$A \ = \ W_{in} \int\limits_{R_c} \left(\frac{N}{M} \right)_{in} \psi_{in}(x) \ dx \tag{2.92}$$

and the removal rate by

$$Q \ = \ \sum_j W_{out \ j} \int\limits_{R_c} \left(\frac{N}{M} \right)_j \psi_{out \ j}(x) \, dx \tag{2.93}$$

In equation (2.93), j indexes the different physical output streams. The integral over the surface of the reference region is inconvenient and this can be converted to an integral over the volume by application of the divergence theorem.

$$\int\limits_{S_c} N\psi(x) u.n \ d\sigma \ = \ \int\limits_{R_c} N \ \nabla.u \, \psi(x) \, dx \tag{2.94}$$

The final working equation for steady-state operation is

$$N \int\limits_{R_c} \nabla.u\,\psi(x)\,dx + \int\limits_{R_c} \frac{R(\psi(x),x,F[\psi(x)])}{\bar{m}(x)} \ dx$$

$$- \int\limits_{R_c} \frac{1}{\bar{m}(x)} \int\limits_{R'(x)} R(\psi(x'),x',F[\psi(x)]) b(x;x') \, dx' \, dx$$

$$+ \int\limits_{R_c} \frac{N}{\bar{m}(x)} \int\limits_{R''(x)} \psi(x') u(x').\nabla\bar{m}(x') a(x;x') \ dx' \, dx$$

$$= - \sum_j W_{outj} \left(\frac{N}{M} \right)_j \int\limits_{R_c} \psi_{outj}(x) dx \ + \ W_{in}\left(\frac{N}{M} \right)_{in} \int\limits_{R_c} \psi_{in}(x) \ dx \tag{2.95}$$

This equation can be specialized in a number of ways to suit the requirements of any particular application. It can be written in terms of mass density functions by substituting for $\psi(x)$ in terms of $p(x)$ using equations such as (2.79).

Because the region R_c is arbitrary, equation (2.95) can be written as a functional integro-differential equation.

$$N\nabla.\pmb{u}\ \psi(\pmb{x})\ +\ \frac{R\big(\psi(\pmb{x}),\pmb{x},F\,[\psi(\pmb{x})]\big)}{\bar{m}(\pmb{x})}$$

$$-\ \frac{1}{\bar{m}(\pmb{x})}\int\limits_{R'(\pmb{x})} R\big(\psi(\pmb{x}'),(\pmb{x}'),F\,[\psi(\pmb{x}')]\big)b(\pmb{x};\pmb{x}')d\pmb{x}'$$

$$+\frac{N}{\bar{m}(\pmb{x})}\int\limits_{R''(\pmb{x})} \psi(\pmb{x}')\pmb{u}(\pmb{x}').\nabla\bar{m}(\pmb{x}')\ a(\pmb{x};\pmb{x}')d\pmb{x}'$$

$$=-\sum\ W_{outj}\left(\frac{N}{M}\right)_j \psi_{outj}(\pmb{x})\ d\pmb{x}\ +\ W_{in}\left(\frac{N}{M}\right)_{in}\psi_{in}(\pmb{x})$$

(2.96)

This form of the population balance equation is usually the most convenient for analytical solution.

The population balance equation must be solved subject to the condition

$$\int\ \psi(\pmb{x})\ d\pmb{x}\ =\ 1$$

(2.97)

where the integral is taken over the entire phase space.

The two breakage functions $b(\pmb{x};\pmb{x}')$ and $a(\pmb{x};\pmb{x}')$ must satisfy two important conditions.

$$\int\limits_{R_{A'}} b(\pmb{x};\pmb{x}')d\pmb{x}\ =\ 1$$

$$\int\limits_{R_{A''}} a(\pmb{x};\pmb{x}')d\pmb{x}\ =\ 1$$

(2.98)

$R_{A'}$ and $R_{A''}$ are the regions of the phase space that can be reached by progeny particles that are formed by breakage (for *b*) or attrition (for *a*) at point $\pmb{x}'$. These are called the accessible regions of phase space and they are complementary to the feeder regions R' and R'' The determination of the feeder regions R' and R'' and the accessible regions $R_{A'}$ and $R_{A''}$ can be quite difficult and we will be looking at some specific cases later. It is usually easier to define the the accessible regions than the feeder regions and this fact can have a significant effect on the on the choice of computational algorithms that can be used successfully.

In most applications the fundamental population balance equation will be reduced to an appropriate discrete form with the region R_c corresponding to the appropriate particle class.

2.13   The General Population Balance Equation for Comminution Machines

The population balance equation provides a powerful model for the description of industrial comminution machines. It allows the development of a uniform model that describes the operating behavior of rod, ball, semi-autogenous, and

autogenous mills. Because of the great importance of these operations in practice some effort is devoted here to the careful development of the models from the detailed population balance equation (2.96).

A useful restricted form of the general population balance equation is generated under the following conditions: only one internal coordinate, the particle size, and it is assumed that the breakage and wear processes are not dependent on the position of the particle in the mill so that the external coordinates are irrelevant. The general equation (2.96) is accordingly written

$$N\frac{d}{dx}(u(x)\psi(x)) + \frac{R(\psi(x), x, F[\psi(x)])}{\beta x^3}$$

$$- \frac{1}{\beta x^3}\int_{R'(x)} R(\psi(x'), x', F[\psi(x')])b(x;x')dx'$$

$$+ \frac{N}{\beta x^3}\int_{R''(x)} \psi(x')u(x')\frac{d\beta x'^3}{dx}a(x;x')\ dx'$$

$$= -W\left(\frac{N}{M}\right)_{out}\psi_{out}(x) + W\left(\frac{N}{M}\right)_{in}\psi_{in}(x)$$

(2.99)

In equation (2.99) the scalar x represents the particle size d_p and the average mass of a particle is related to the size by

$$\bar{m}(x) = \beta x^3 \qquad (2.100)$$

A commonly-used model for the rate of attrition and other wear processes such as chipping is that this rate is proportional to the surface area of the particle. Using a spherical particle as a model

$$\frac{\pi}{6}\frac{dx^3}{dt} = -\frac{\kappa'\pi x^2}{2} \qquad (2.101)$$

which implies that the velocity at which a wearing particle moves in phase space is constant and is given by

$$u(x) = \frac{dx}{dt} = -\kappa' \qquad (2.102)$$

A more general model for surface wear processes is

$$\frac{dx}{dt} = -\kappa(x) = -\kappa x^\Delta$$

where Δ is a constant between 0 and 1.
This implies that

$$\frac{dm}{dt} = -\kappa\frac{\pi\rho_s x^{2+\Delta}}{2} \qquad (2.104)$$

If $\Delta = 0$ the specific surface wear rate is constant while $\Delta > 0$ means that the specific surface wear rate increases as the size of the particle increases. As shown above $\Delta = 0$ is equivalent to the assumption that surface wear rate is proportional to the surface area of the particle while $\Delta = 1$ means that the surface wear rate is proportional to the mass of the particle.

It is usual to develop the population balance equation in terms of the distribution by mass and equation (2.99) is converted using equation (2.79)

$$\psi(x) = \frac{M}{N}\frac{p(x)}{\beta x^3}$$

(2.105)

Equation (2.99) becomes

$$-\frac{M}{\beta}\frac{d(\kappa(x)p(x)/x^3)}{dx} + \frac{R(p(x),x,F[p(x)])}{\beta x^3}$$

$$-\frac{1}{\beta x^3}\int_{R'(x)} R(p(x'),x',F[p(x')])b(x;x')dx' - \frac{M}{\beta x^3}\int_{R''(x)} \frac{p(x')}{x'^3}3\kappa(x')x'^2 a(x;x')dx'$$

(2.106)

$$= -W\frac{p_{out}(x)}{\beta x^3} + W\frac{p_{in}(x)}{\beta x^3}$$

The rate of destruction is intensive with respect to the average mass density over phase space. Consider two systems having the identical phase space but containing different total masses. In particular $\psi(x)$ is identical for both systems. Then

$$\frac{R_1(p(x),x,F[p(x)])}{R_2(px),x,F[p(x)])} = \frac{M_1}{M_2}$$

(2.107)

Therefore

$$R(p(x),x,F[p(x)]) = MR'(p(x),x,F[p(x)])$$

(2.108)

Furthermore a logical assumption relating to the physical breakage process would suggest that

$$R(p(x),x,F[p(x)]) = Mp(x)k(x,F[p(x)])$$

(2.109)

$k(x,F[p(x)])$ is the specific rate of breakage of material of size x and it represents the breakage rate when 1kg of material of size x is in the mill.

The average residence time in the mill is

$$\tau = \frac{M}{W}$$

(2.110)

where W is the flowrate through the mill.

Equation (2.106) becomes

$$-\tau\frac{d\kappa(x)p(x)}{dx} \;+\; 3\tau\kappa(x)\frac{p(x)}{x} \;+\; \tau R(p(x),x,F[p(x)]$$

$$-\tau\int_{R'(x)} R'(p(x'),x'F[p(x')])b(x;x')dx' \;-\; \tau\int_{R''(x)} 3\kappa(x')\frac{p(x')}{x'}a(x;x')dx' \qquad (2.111)$$

$$= p_{in}(x) \;-\; p_{out}(x)$$

This is the fundamental population balance equation for any comminution process and it incorporates both autogeneous and media-induced breakage. It is difficult to solve this integro-differential equation principally because the functions R, b and a are strongly nonlinear. Under conditions that apply to practically useful situations, only numerical solutions are possible. These will be developed for practical situations in Chapter 5.